UNIT 3 : SEQUENCE

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3.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- sequence, subsequence, bounded sequence, monotonic sequence
- convergent sequence, divergent sequence, algebra of convergent sequences
- cauchy sequence, Cauchy's General Principle of Convergence.
- cantor's Theorem on nested intervals.

3.2 INTRODUCTION

The notion of a sequence of real numbers is required to obtain the sum of an infinite number of real numbers. In this unit we will introduce the notion of sequence, subsequence and discuss the convergence,

boundedness, monotonicity of sequence. We will also discuss Cantor's theorem on nested intervals.

3.3 SEQUENCE

A function f from the set N of natural numbers to the set R of real numbers is called a real sequence, i.e., $f : N \to R$ is a real sequence.

Note : 1. Since we will deal with real sequences only, we will omit the adjective real.

2. f (1), f (2), f (3), ... are the images under f. f (n) is called the nth term (or element) of the sequence.

3. The sequence is, generally, denoted by { f (1), f (2), f (3),.....} or { f (n)} or $\{u_n\}$ or $\{s_n\}$, where $u_n = s_n = f(n)$.

4. The set consisting of all distinct elements of a sequence, without repetition, and without regard to the position of a term, is called the range of the sequence.

Examples :

1. Let $f : N \to R,\ f(n) = \frac{n}{n+1},\ n \in N$

Here $u_1 = \frac{1}{2},\ u_2 = \frac{2}{3},\ u_3 = \frac{3}{4}, \ldots\ldots\ldots$

Thus, the sequence is $\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\ldots \right\}$ i.e. $\left\{ \frac{n}{n+1} \right\}$

2. Let $f : N \to R,\ f(n) = (-1)^{n+1},\ n \in N$

Here $u_1 = 1,\ u_2 = -1,\ u_3 = 1,\ u_4 = -1\ldots\ldots\ldots$

Thus, the sequence is { 1, –1, 1, –1, ...}, i.e. $\left\{ (-1)^{n+1} \right\}$

Note :The range of the sequence is { –1, 1}.

3. Let $f : N \to R,\ f(n) \begin{cases} = -1, \text{ if } n = 1, 2 \\ = f(n-1) + f(n+2), \text{ if } n > 2 \end{cases}$

Here $u_1 = 1,\ u_2 = 1$

$u_3 = u_2 + u_1 = 2,\ u_4 = u_3 + u_2 = 3,$

$u_5 = u_4 + u_3 = 5,\ \ldots\ldots\ldots$

Thus, the sequence is { 1, 1, 2, 3, 5, 8, ...}

Note : The above sequence is known as Fibonacci sequence after the name of the Italian mathematician Fibonacci aka Leonardo Fibonacci (c 1170 – c 1250). Fibonacci was instrumental in popularising Decimal place Value System (developed in India) in through his book Liber Abaci.

4. Let $f : N \to R,\ f(n) = 1,\ n \in N$

Here $u_1 = u_2 = u_3 = \ldots\ldots\ldots = 1$

The sequence is { 1, 1, 1, ...}

Note : The above sequence is called a constant sequence.

3.3.1 Subsequence

Let us consider the following two sequences,

{ 1, 2, 3, 4, 5, ...} (1)

and {1, 3, 6, 10, 15, 21, ...} (2)

We note that the sequence (2) has been obtained from the sequence (1) by omitting some terms of (1) and without disturbing the relative positions of the remaining terms. In this case, the sequence (2) is called a subsequence of the sequence (1).

A subsequence of a sequence is defined as follows.

Let $\{u_n\}$ be a sequence. Let n_1, n_2, n_3 ..... be natural numbers such that $n < n_2 < n_3 < \ldots\ldots\ldots$ Then the sequence $\{u_{n1}, u_{n2}, u_{n3}, \ldots\ldots\}$ i.e. $\{u_{nk}\}$ is called a subsequence of $\{u_n\}$.

Example Let us consider the sequence $f : N \to R,\ f(n) = \frac{1}{n},\ n \in N$

i.e. the sequence is $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\ldots\ldots\right\}$

The sequence $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\ldots\ldots\right\}$ is a subsequence of $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\ldots\ldots\right\}$.

3.3.2 Bounded Sequence

A sequence $\{u_n\}$ is said to be bounded above if there exists a number

M such that $u_n \le M,\ \forall n \in N$. M is said to be an upper bound of the sequence.

A sequence $\{u_n\}$ is said to be bounded below if there exists a number m such that $u_n \le m,\ \forall n \in N$. m is said to be a lower bound of the sequence.

A sequence is said to be bounded if it is both bounded above and bounded below, i.e. if $m \le u_n \le M,\ \forall n \in N$

The least upper bound (supremum) of the range is called the least upper bound of the sequence, whereas the greatest lower bound (infimum) of the range is called the greatest lower bound of the sequence.

Examples

1. Let $f : N \to R,\ f(n) = \frac{1}{n},\ n \in N$

Here the sequence is $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\ldots\right\}$

lub = 1, glb = 0

The sequence is bounded

2. Let $f : N \to R,\ f(n) = (-1)^{n+1},\ n \in N$

lub = 1, glb = –1

3. Let $f : N \to R,\ f(n) = 2^n,\ n \in N$

The sequence is { 2, 2^2, 2^3, 2^4, ...}

glb = 2

There exists no upper bound.

The sequence is bounded below.

4. Let $f : N \to R,\ f(n) = -3n,\ n \in N$

The sequence is { –3, –6, –9, –12, ...}

lub = –3

There exists no lower bound.

The sequence is bounded above.

3.3.3 Monontonic Sequence

Let us consider the sequence $f : N \to R,\ f(n) = \frac{1}{n},\ n \in N$

The sequence is $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\right\}$

We have $\left\{1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \ldots\ldots\right\}$

such a sequence is called a monotonically decreasing sequence.

Again, consider the sequence $f : N \to R,\ f(n) = 2^n,\ n \in N$

The sequence is { 2, 2^2, 2^3, 2^4, ...}

We note that $2 < 2^2 < 2^3 < 2^4 < .....$

such a sequence is called monotonically increasing sequence.

Definition Let $\{u_n\}$ be a sequence $\{u_n\}$ is said to be a monotonically decreasing sequence if $u_1 \geq u_2 \geq u_3 \geq u_4 \geq .....$

$\{u_n\}$ is said to be a monotonically increasing sequence if $u_1 \leq u_2 \leq u_3 \leq u_4 \leq .....$

A sequence which is either monotomically increasing or monotomically decreasing is called a monotonic sequence.

Note : Constant sequences are the only sequences which are both monotonically increasing and monotonically decreasing.

Do You Know ?

Let $a_1, a_2, ...a_n$ be n positive real numbers.

The arithmetic mean (AM) is defined by $\frac{a_1 + a_2 + ... + a_n}{n}$

The geometric mean (GM) is defined by $\sqrt[n]{a_1 a_2 ..... a_n}$

AM $\geq$ GM

The equality dolds good if $a_1 = a_2 = ...= a_n$.

Example 1. Show that the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ is monotonically increasing.

Solution Here $u_n = \left(1 + \frac{1}{n}\right)^n$

$$\therefore u_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

Let us consider the (n + 1) positive numbers

$1, 1+\frac{1}{n}, 1+\frac{1}{n}, ..., 1+\frac{1}{n}$ (n times)

We know that AM > GM.

$$\therefore \frac{1+\left(1+\frac{1}{n}\right)+\left(1+\frac{1}{n}\right)+\left(1+\frac{1}{n}\right)}{n+1}$$

$$> \left(1.\ (1+\frac{1}{n})\ \ (1+\frac{1}{n})\ .....(1+\frac{1}{n})\right)^{\frac{1}{n+1}}$$

or, $\frac{1+n\left(1+\frac{1}{n}\right)}{n+1} > \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$

or, $\left(\frac{1}{n+1}+1\right) > \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$

or, $\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$

i.e. $u_{n+1} > u_n, \forall n \in N$

Thus $u_1 < u_2 < u_3 < u_4 <$ .....

is a monotomically increasing sequence.

Theorem 3.1 Every subsequence of a monotonic sequence is also monotonic.

Proof : Let $\{u_n\}$ be a monotonic increasing sequence. Then $u_1 \le u_2 \le u_3 \le u_4 \le .....$

Let $\{u_{n_k}\}$ be a sebsequence of $\{u_n\}$. Then $n_1 < n_2 < n_3 <$ .....

Hence $u_{n_1} < u_{n_2} < u_{n_3} <$ .....

$\therefore \{u_{n_k}\}$ is monotonic increasing similarly, we can prove for a monotomic decreasing sequence.

3.4 CONVERGENCE OF A SEQUENCE

First we define limit of a sequence.

Let $\{u_n\}$ be a sequence. A real number is said to be a limit of $\{u_n\}$ if for every $\varepsilon > 0$, there exists a positive integer m, which depends upon the choice of ε, such that for every $n \in \underline{\underline{N}}$, $|u_n - L| < \varepsilon$, for all $n \geq m$.

i.e. $L - \varepsilon < u_n < L + \varepsilon$, for all $n \geq m$. Now we define a convergent sequence.

A sequence $\{u_n\}$ is said to be convergent if it has a limit $L \in R$. We say that the sequence converges to L. We write $\lim u_n = L$ or $\lim_{n \to \infty} u_n = L$

Note : A sequence is said to be divergent if it is not convergent.

Example 2 : Examine the convergence of the sequence $\left\{\frac{1}{n}\right\}$

Solution : Intuitively we have $\lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ The sequence converges to 0.

Now, let us apply the definition to examine the convergence.

We choose $\varepsilon > 0$.

We know, by Archimedean property of real numbers, that there exists a positive integer m such that $0 < \frac{1}{m} < \varepsilon$.

Thus, $0 < \frac{1}{n} < \varepsilon, \ \forall n \geq m$

$\therefore \ |\frac{1}{n} - 0| < \varepsilon, \ \forall n \geq m$

$\therefore \lim_{n \to \infty} \frac{1}{n} = 0$

$\therefore$ The sequence converges to 0.



CHECK YOUR PROGRESS

Q.1: Examine the convergence of the sequence $\left\{\frac{1}{n^p}\right\},\ p>0$

Q.2: Examine the convergence of the sequence $\left\{\frac{2n+1}{n+1}\right\}$.

Theorem 3.2. If a sequence is convergent, then its limit is unique.

Proof : If possible, let a sequence $\{u_n\}$ converge to two distinct limits, say L_1 and L_2.

Then given $\varepsilon > 0$, there exists a positive integer m such that

$$|u_n - L_1| < \frac{1}{2}\varepsilon,\ \forall n \geq m$$

$$|u_n - L_2| < \frac{1}{2}\varepsilon,\ \forall n \geq m$$

Now $|L_1 - L_2| = |L_1 - u_n + u_n - L_2|$

$$\leq |L_1 - u_n| + |u_n - L_2|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

$\therefore$ $|L_1 - L_2|$ is less than any positive ε, however small.

This $\Rightarrow$ $L_1 - L_2 = 0$, i.e. $L_1 = L_2$.

Thus, the limit is unique.

Theorem 3.3 A convergent sequence is bounded.

Proof : Let the sequence $\{u_n\}$ converge to L.

Then given $\varepsilon > 0$, there exists a positive integer m such that

$$|u_n - L| < \varepsilon,\ \forall n \geq m$$

$$\Rightarrow L - \varepsilon < u_n < L + \varepsilon,\ \forall n \geq m$$

Let b = min $\{l - \varepsilon,\ u_1,\ u_2,\ u_3, \ldots\ldots,\ u_{m-1}\}$

B = max $\{l + \varepsilon,\ u_1,\ u_2,\ u_3, \ldots\ldots,\ u_{m-1}\}$

$\therefore b \leq u_n \leq B,\ \forall n$

$\therefore \{u_n\}$ is bounded.

Note 1. We have see that if a sequence is convergent, then it is bounded contrapositively, if a sequence is not bounded, then it is not convergent.

2. The converse of Theorem 3.3 is not true, i.e., if a sequence is bounded, then it may not be convergent.

Let us consider the sequence $\{(-1)^n\},\ n \in N$

Clearly, the sequence is bounded.

lub = 1, glb = 1

If possible, let $\{(-1)^n\}$ converge to L. i.e. $\lim_{n \to \infty} (-1)^n = L$

We choose ε=1. Then there exists a positive integer m such that

$$|(-1)^n - L| < 1,\ \forall n \geq m$$

This $\Rightarrow |(-1)^{2m} - L| < 1,$ and $|(-1)^{2m+1} - L| < 1$

i.e. $|1 - L| < 1,$ and $|-1 - L| < 1$

i.e. $|1 - L| < 1,$ and $|-1 + L| < 1$

$$\therefore 2 = |1 - L + 1 + L|$$
$$\leq |1 - L| + |1 + L|$$
$$< 1 + 1 = 2$$

which is not possible

$\therefore$ our assumption is wrong.

$\therefore \{(-1)^n\}$ is not convergent.

Theorem 3.4 A monotonic increasing sequence, if bounded above, is convergent, and it converges to the least upper bound.

Proof : Let $\{u_n\}$ be a monotonic increasing sequence, bounded above.

Let B be the lub.

$$\therefore u_n \leq B,\ \forall n \in N$$

For a given $\varepsilon > 0$, there exists a natural member m such that $u_m > B - \varepsilon$

$\because \{u_n\}$ is a monotomic increasing sequence,

$\therefore B - \varepsilon < u_m \leq u_{m+1} \leq u_{m+2} \leq \ldots\ldots\ldots \leq B$

i.e. $B - \varepsilon < u_n \leq B + \varepsilon,\ \forall n \geq m$

$\therefore$ The sequence is convergent, and it converges to B.

Theorem 3.5 A monotonic decreasing sequence, if bounded below, is convergent and it converges to the greatest lower bound.

Proof : Proceed as above.



CHECK YOUR PROGRESS 2

Q.1: Let the sequence $\{u_n\}$ converge to L. Show that the sequence $\{|u_n|\}$ to also converges to L. Is the converse true? Justify your answer.

Q.2: Let the sequence $\{u_n\}$ converge to L, and $u_n \le B$, for every n. Show that $L \le B$.

Q.3: Let the sequence $\{u_n\}$ converge to L. Show that every subsequence of $\{u_n\}$ also converges to L.

3.4.1 Algebra of Convergent Sequences

Definition Let $\{u_n\}$, and $\{u_n\}$ be two sequences of real numbers. Let c be a real number.
Then

(i) $\{u_n\} \pm \{\upsilon_n\} = \{u_n \pm \upsilon_n\}$

(ii) $c\{u_n\} = \{cu_n\}$

(iii) $\{u_n\} = \{\upsilon_n\} = \{u_n \upsilon_n\}$

(iv) $\left\{\frac{u_n}{\upsilon_n}\right\} = \left\{\frac{u_n}{\upsilon_n}\right\}$, provided $\{u_n\}$ is a sequence of non zero real numbers.

Theorem 3.6 Let $\{u_n\}$ and $\{u_n\}$ be two convergent sequences converging to u and v respectively. Then

(i) $\lim (u_n + \upsilon_n) = u + \upsilon$

(ii) $\lim (cu_n) = cu$, where c is a real number

(iii) $\lim u_n \upsilon_n = u\upsilon$

(iv) $\lim u_n \upsilon_n = \frac{u}{\upsilon}$, provided $\{v_n\}$ is a sequence of non zero real numbers, and $\upsilon \neq 0$

Proof : (i) $\because \{u_n\}$ converges to u,

$\therefore$ given $\varepsilon > 0$, there exists a positive integer m, such that

$$|u_n - u| < \frac{1}{2}\varepsilon, \ \forall n \ge m_1$$

Again, $\because \{\upsilon_n\}$ converges to υ,

$\therefore$ given $\varepsilon > 0$, there exists a positive integer m_2 such that

$$|\upsilon_n - \upsilon| < \frac{1}{2}\varepsilon, \ \forall n \geq m_2$$

Now $|(u_n + \upsilon_n) - (u + \upsilon)|$

$= |(u_n - u) + (\upsilon_n - v)|$

$\leq |u_n - u| + |\upsilon_n - \upsilon|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon, \ \forall n \geq m$

where $m = \max\{m_1, m_2\}$

$\therefore \{u_n + \upsilon_n\}$ converges to u + υ

i.e. $\lim(u_n + \upsilon_n) = u + \upsilon$

ii) If c = 0, there is nothing to prove we assume that c $\neq$ 0.

We have $|cu_n - cu| = |c||u_n - u|$ (1)

$\therefore \{u_n\}$ converges to u, given $\varepsilon > 0$, there exists a positive integer m such that

$$|u_n - u| < \frac{\varepsilon}{|c|}, \ \forall n \geq m \quad (2)$$

$\therefore$ from (1) and (2),

$\therefore |cu_n - cu| = < \varepsilon, \ \forall n \geq m$

$\therefore \{cu_n\}$ converges to cu,

i.e. $\lim cu_n = cu$

(iii) We have $|u_n + \upsilon_n - u\upsilon|$

$= |u_n(\upsilon_n - \upsilon) + \upsilon(u_n - u)|$

$\leq |u_n||\upsilon_n - \upsilon| + |\upsilon||u_n - u|$ (1)

We know that a convergent sequence is bounded. $\therefore \{u_n\}$ is convergent, so it is bounded.

$\therefore$ There exists a positive number B_i such that $|u_n| < B$, for all $n \in N$ (2)

Let k = max { B, $|\upsilon|$} (3)

From (1), (2), and (3)

$|u_n\upsilon_n - u\upsilon|$

$< k \ |\upsilon_n - \upsilon| + k|u_n - u|$ (4)

Since $\{u_n\}$, and $\{\upsilon_n\}$ converge to u and υ respectively. Therefore, for every $\varepsilon > 0$, there exist positive integers m_1 and m_2 such that

$|u_n - u| < \frac{\varepsilon}{2k}$, for all $n \geq m_1$

$|\upsilon_n - \upsilon| < \frac{\varepsilon}{2k}$, for all $n \geq m_2$

Let m = max $\{m_1, m_2\}$

$|u_n - u| < \frac{\varepsilon}{2k}$, for all $n \geq m$

$\therefore |\upsilon_n - \upsilon| < \frac{\varepsilon}{2k}$, for all $n \geq m$

$\therefore$ from (4)

$|u_n\upsilon_n - u\upsilon| < k\frac{\varepsilon}{2k} + k\frac{\varepsilon}{2k} = \varepsilon$, for all $n \geq m$

$\therefore \{u_n\upsilon_n\}$ converges to $u\upsilon$

i.e., $\lim u_n\upsilon_n = u\upsilon$

(iv) First, we will prove that there exists a positive integer m such that

$|\upsilon_n| > \frac{1}{2}|\upsilon|$, for all $n \geq m_1$

We have $\upsilon = \upsilon - \upsilon_n + \upsilon_n$

$\therefore |\upsilon| = \upsilon - \upsilon_n + \upsilon_n|$

Now we can choose m_1 so that

$|\upsilon - \upsilon_n| = |\upsilon_n - \upsilon|$

$< \frac{1}{2}|\upsilon|$, for all $n \geq m_1$ $(\because \lim \upsilon_n = \upsilon)$

$\therefore$ from (1), $|\upsilon| < \frac{1}{2}|\upsilon| + |\upsilon_2|$

or, $\frac{1}{2}|\upsilon| < |\upsilon_n|$

or $|\upsilon_n| > \frac{1}{2}|\upsilon|$, for all $n \geq m_1$

or, $\frac{1}{|\upsilon_n|} < \frac{1}{\frac{1}{2}|\upsilon|}$ for all $n \geq m_1$ (1)

Next, we prove that $\lim \frac{1}{\upsilon_n} = \frac{1}{\upsilon}$

Since $\lim \upsilon_n = \upsilon$, $\therefore$ given $\varepsilon > 0$, there exists a positive integer m_2 such that

$|\upsilon_n - \upsilon| < \frac{1}{2}|\upsilon|^2 \varepsilon$, for all $n \geq m$

Let, m = max $\{m_1, m_2\}$

Now $\left|\frac{1}{\upsilon_n} = \frac{1}{\upsilon}\right| = \left|\frac{\upsilon - \upsilon_n}{\upsilon_n \upsilon}\right|$

$$= \frac{|\upsilon_n - \upsilon|}{|\upsilon_n||\upsilon|}$$

$$< \frac{\frac{1}{2}|\upsilon|^2 \varepsilon}{|\upsilon||\frac{1}{2}\upsilon|} = \varepsilon, \text{ for all } n \geq m$$

$$\therefore \lim \frac{1}{\upsilon_n} = \frac{1}{\upsilon}$$

Now, we prove that $\lim \frac{u_n}{\upsilon_n} = \frac{1}{\upsilon_n}$

We have $\lim \frac{u_n}{\upsilon_n} = \lim \left(u_n \frac{1}{\upsilon_n}\right)$

$$= (\lim u_n)\ (\lim \frac{1}{\upsilon_n}), \text{ By (iii)}$$

$$= u\frac{1}{\upsilon} = \frac{u}{\upsilon}$$

$\therefore \left\{\frac{u_n}{\upsilon_n}\right\}$ converges to $\frac{u}{\upsilon}$

i.e. , $\lim \frac{u_n}{\upsilon_n} = \frac{u}{\upsilon}$



CHECK YOUR PROGRESS 3

Q. 1: Write dow two divergent sequences $\{u_n\}$ and $\{\upsilon_n\}$ such that $\{u_n + \upsilon_n\}$ is convergent.

Q. 2: Write down one divergent sequence $\{u_n\}$ and a convergent sequence $\{u_n\}$ such that $\{u_n \upsilon_n\}$ is convergent

Do you Know ?

Sandwich is an item of food consisting of two pieces of bread with a filling between them. It is said to be named after the fourth Earl of Sandwich, John Montagu (1718-1792) "who ate a snack of this kind in order not to have to leave the gaming table." He was not a mathematician.

Another meaning : (noun) anything in a similar arrangement (adj) of the nature of, relating to or resembling, a sandwich (verb) to lay or place between two layers.

3.5 LIMIT POINT OF A SEQUENCE

A real number L is said to be a limit point of a sequence $\{U_n\}$, if every neighbourhood of L contains an infinite number of members of the sequence,

i.e., L is a limit point of $\{U_n\}$ if given $\in>0$, however small,

$U_n \in]L-\in, L+\in[$, for an infinite number values of n, i.e.,

$|U_n - L| < \in$, for an infinite number of values of n.

Clearly, L is not a limit point of $\{U_n\}$, if there exists $\in>0$ such that $U_n \in]L-\in, L+\in[$ for at most a finite number of values of n.

Note :

1. A convergent sequence has only one limit point, its limit.
2. Any bounded sequence has at least one limit point.
3. If a baunded sequence is not convergent. Then it has at least two limit points.
4. A limit point of the range set of a sequence is also a limit point of the sequence.

   The converse of the above result is not necessarily true.

Example : Let us consider the sequence $\{U_n\}$, where $U_n = 1$.

For every $\in>0$, $U_n = 1 \in]1-\in, 1+\in[$, $\forall\, n \in N$

$\therefore$ 1 is a limit point of the sequence.

Let $\alpha \in R$, and $\alpha \neq 1$. Then for all n, $|U_n - \alpha| = |1-\alpha| \not< \in$.

Thus, there is no limit point other than 1, i.e., 1 is the only limit point

of the sequence.

We note that 1 is not a limit point of the range {1}. The range {1} has no limit point.

Example : Let us consider the sequence $\{U_n\}$, where $U_n = \frac{1}{n}$.

For every $\in > 0$, $U_n = \frac{1}{n} \in]\, 0 - \in, 0 + \in [$, $\forall\, n \in \underline{N}$

$\therefore$ 0 is a limit point point of the sequence.

We note that 0 is also a limit of the range $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right\}$

Example : Let us consider the sequence $\{U_n\}$, where $U_n = 1+(-1)^n$.

The sequence is

{ 0, 2, 0, 2, 0, 2, ... }

0 and 2 are the limit points of the sequence.

The range {0, 2} has no limit point.

Example : Let us consider the sequence $\{U_n\}$, where $U_n = (-1)^n$, $n \in \underline{N}$.

The sequence is {–1, 1, –1, 1, –1, 1, ...}

–1, and 1 are the limit points of the sequence.

The range {–1, 1} has no limit point.

Theorem : (Bolzano - Weierstrass) Every bounded sequence has a limit point.

Proof : Let $\{U_n\}$ be a bounded sequence.

Let $A = \{U_n : n \in N\}$ be the range.

Since $\{U_n\}$ is bounded, A is also bounded.

Now, there are two possibities,– either A is finite or A is infinite.

Case 1 : Let A be finite.

Then there must exist at least one member $\alpha \in A$ such that Un = α for an infinite number of values of n.

Thus, every neighbourhood $]\alpha - \in, \alpha + \in [$, $\in > 0$, of contains $Un (= \alpha)$ for an infinite number of values of n.

$\therefore$ α is a limit point of the sequence {Un}.

Case 2 : Let A be infinite.

Now, A is bounded.

$\therefore$ by Bolzano – Weierstrass theorem for sets, A has at least one limit point. Let it be β.

Now, β is a limit point of A.

$\therefore$ every neighbourhood $]\beta-\in, \beta+\in[$, $\in>0$, contains an infinite number of members of A,

i.e., $U_n \in]\beta-\in, \beta+\in[$ for an infinite number of values of n.

$\therefore \beta$ is a limit point of the sequence $\{U_n\}$.

3.6 CANTOR'S THEOREM ON NESTED INTERVALS

Let us consider the intervals $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, $I_3 = [a_3, b_3]$, ... such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq .....$, such intervals are called nested intervals.

The family $\{I_n : n \in N\}$ is called to be a family of nested intervals.

Example Let us consider the intervals

$I_1 = [-1,1]$, $I_2 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $I_3 = \left[-\frac{1}{3}, \frac{1}{3}\right]$, $I_4 = \left[-\frac{1}{4}, \frac{1}{4}\right]$,..... clearly

$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq .....$

$\left\{\left[-\frac{1}{n}, \frac{1}{n}\right] : n \in N\right\}$ is a family of nested intervals.

Theorem 3.7. (Sandwich Theorem)

Let $\{u_n\}$, $\{\upsilon_n\}$, $\{w_n\}$ be three sequences of real numbers, and m be a positive interger such that $u_n \le \upsilon_n \le w_n$, for all $n \ge m$

If $\{u_n\}$, $\{w_n\}$ converge to L, $\{\upsilon_n\}$ will also converge to L.

Proof The sequences $\{u_n\}$, $\{w_n\}$ converge to L.

$\therefore$ given $\varepsilon > 0$, there exist positive integers m_1 and m_2 such that

$|u_n - L| < \varepsilon$ for all $n \ge m_1$

$|w_n - L| < \varepsilon$ fro all $n \ge m_2$

Let $m_3 = \max\{m_1, m_2\}$

Then $L - \varepsilon < u_n < L + \varepsilon$ for all $n \ge m_3$

$L - \varepsilon < w_n < L + \varepsilon$ for all $n \ge m_3$

Let $k = \max\{m_3, m\}$

Then $L - \varepsilon < u_n < \upsilon_n < w_n < L + \varepsilon$ for all $n \ge k$

i.e. $|\upsilon_n - L| < \varepsilon$ for all $n \geq k$

The sequence $\{\upsilon_n\}$ converges to L.

Note the sequence $\{\upsilon_n\}$ is sandwiched between $\{u_n\}$, $\{w_n\}$.

Theorem 3.8 Cantor's Theorem on nested intervals

Let $\{[a_n, b_n]\}$ be a sequence of closed and bounded intervals such that

(i) $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$, for all $n \in N$

(ii) $\lim(b_n - a_n) = 0$

Then $\bigcap_{n=1}^{\infty}$ contains exactly one point.

Proof : We have

$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq .....$

$\therefore a_n \leq a_{n+1}, b_{n+1} \leq b_n$ for all $n \in N$

Again $a_n \leq b_n$, for all $n \in N$

$\therefore a_1 \leq a_2 \leq ... \leq a_n \leq ... \leq b_n \leq .... \leq b_2 \leq b_1$

$\therefore \{a_n\}$ is a monotonic increasing sequence, bounded above.

$\therefore \{a_n\}$ is convergent (Thm 3.4)

Agian, $\{b_n\}$ is a monotonic decreasing sequence, bounded below.

$\therefore \{b_n\}$ is convergent (Thm 3.5)

Let lim an = a, lim b_n = b

Now lim (b_n–a_n) = 0, given

$\therefore$ b – a = 0

i.e. b = a = s, say.

$\therefore$ s is the least upper bound of the sequences $\{a_n\}$, and the greatest lower bound of the sequence $\{b_n\}$.

$\therefore$ $a_n \leq s$, and $s \leq b_n$ for all $n \in N$

$\therefore$ $s \in [a_n - b_n]$ for all $n \in N$

i.e. $s \in \bigcap_{n=1}^{\infty} [a_n - b_n]$

Uniqueness If possible, let $t \in \bigcap_{n=1}^{\infty} [a_n - b_n]$ then $a_n \le t \le b_n$ for all $n \in N$

We define a sequence $\{u_n\}$, where u_n – t for all $n \in N$

$\therefore$ lim u_n = t

Now $a_n \le u_n \le b_n$ for $n \ge 1$

Also $\lim a_n = \lim b_n = s$

$\therefore$ by Sandwich Theorem, lim u_n = s

Thus, t = s

Note : 1. A nested sequence of open and bounded intervals may not have a nonempty intersection.

Let us consider the nested sequence of open intervals

$$]0, 1[\supseteq]\, 0, \frac{1}{2}[\supseteq]\, 0, \frac{1}{3}[\supseteq]\, 0, \frac{1}{4}[\supseteq .....$$

Each of these intervals is bounded.

But $\bigcap_{n=1}^{\infty}]\, 0, \frac{1}{n}[= \phi$

2. A nested sequence of closed and unbounded intervals may not have a nonempty intersection.

Remark : Georg Cantor (1895–1918) was a celebrated mathematician. He is regarded as the father of Set Theory.

3.7 CAUCHY'S GENERAL PRINCIPLE OF CONVERGENCE

Theorem 3.9 A sequence $\{u_n\}$ is said to be convergent if given $\varepsilon > 0$, there exists a positive integer m such that $|u_n - u_k| < \varepsilon$ where n > m, k > m, n > k.

Proof Let the sequence $\{u_n\}$ converge to u.

Let $\varepsilon > 0$ be given Then $\frac{\varepsilon}{2} > 0$

Now for $\frac{\varepsilon}{2}$, there exists a positive integer m such that $|u_n - u| < \varepsilon/2$ for n > m.

Let n, k > m. Then

$$|u_n - u_k| = |(u_n - u) - (u_k - u)|$$
$$\leq |u_n - u| + |u_k - u|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

i.e. $|u_n - u_k| < \varepsilon$, for all n, k > m

Note : In view of the above theorem, we get the notion of Cauchy sequence after the name of the renowned Frech mathematician Augustin-Louis Cauchy (1789-1857).

Cacuchy Sequence

A sequence $\{u_n\}$ of real numbers is called a Cauchy sequence (or fundamental sequence) if, given $\varepsilon > 0$, there exists a positive integer m such that $|u_n - u_k| < \varepsilon$, for all n> m > k (n>k).

Example Let us consider the sequence $\left\{\frac{1}{n}\right\}$.

Here $u_n = \frac{1}{n}$

For any two positive integers n, and k (n > m, k > m, n > k), we have

$$|u_n - u_k| = \left|\frac{1}{n} - \frac{1}{k}\right| = \left|\frac{k-n}{nk}\right| < \frac{1}{k}$$

Let $\varepsilon > 0$ be given. Taking $k > \frac{1}{\varepsilon}$, we get $\frac{1}{k} < \varepsilon$

$\therefore |u_n - u_k| < \varepsilon$, for all n> m, k > m

$\therefore \{u_n\}$ is a Cauchy sequence.

Theorem 3.10 A Cauchu sequence is bounded.

Proof Let $\{u_n\}$ be a Cauchy sequence. Then given $\varepsilon > 0$, there exists a positive integer m such that

$|u_n - u_k| < \varepsilon$, for n> m, k > m.

We choose ε = 1

$|u_n - u_k| < 1$, for n> m, k > m.

Taking k = m + 1, we get

$|u_n - u_{m+1}| < 1$, for n> m

$\therefore |u_n| \leq |u_{m+1}| + 1$ for n> m

Let B = max $\{|u_1|, |u_2|, \ldots\ldots |u_m|, |u_{m+1}|+1\}$

$\therefore |u_n| \leq B$ for all $n \in N$

$\therefore \{u_n\}$ is bounded.

Do You Know

"Every bounded sequence of real numbers contains a convergent subsequence."

This result is also know as Bolzano-Weierstrass Theorem.

Theorem 3.11 A Cauchy sequence is convergent.

Proof Let $\{u_n\}$ be a Cauchy sequence

$\therefore \{u_n\}$ is bounded.

Therefore, it must have a convergent subsequence. Let it be $\{u_{n_r}\}$

Let it converge to p.

Let $\varepsilon > 0$ be any real number.

Then $\frac{1}{2}\varepsilon > 0$

$\therefore$ There exists a positive integer q such that for any r > q, we have

$$|u_{n_r} - p| < \frac{1}{2}\varepsilon$$

Again $\{u_n\}$ is a Cauchy sequence.

$\therefore$ There exists a positive integer k such that for k>m, and any n > m, we have

$$|u_n - u_k| < \frac{1}{2}\varepsilon$$

Now, for an n > m, we choose r so large that n_r > m, and r > q. Then

$|u_{n_r} - p| < \frac{1}{2}\varepsilon$ is satisfied

Also $|u_n - p| < \frac{1}{2}\varepsilon$ is satisfied for $k = n_r$

Thus, for n > m

$$\begin{aligned} |u_n - p| &= |u_n - u_{n_r} + u_{n_r} - p| \\ &\leq |u_n - u_{n_r}| + |u_{n_r} - p| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \end{aligned}$$

Thus, given $\varepsilon > 0$ there exists m such that for any n > m, $|u_n - p| < \varepsilon$

The Cauchy sequence $\{u_n\}$ converges to p.

Note : Theorems 3.9, 3.10, 3.11 together constitute the Cauchy's General Principle of Convergence which may be stated as follows.

A necessary and sufficient condition forthe convergence of a sequence $\{u_n\}$ is that for every $\varepsilon > 0$, there exists a positive integer m such that

$|u_n - u_k| < \varepsilon$, for n, k > m (n > k).

Do You Know ?

A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

3.8 LIMIT SUPERIOR AND LIMIT INFERIOR OF A SEQUENCE

Let $\{u_n\}$ be a sequence, which is not necessarily bounded.

The limit of superior of the sequence $\{u_n\}$ is defined by

$$\lim_{n\to\infty} \sup u_n = \inf \sup \{u_n, u_{n+1}, u_{n+2}, ...\}$$

We denote limit superior of $\{u_n\}$ by $\overline{\lim}_{n\to\infty} u_n$ or by $\overline{\lim} u_n$

NB : Limit superior, and limit inferior of a real sequence exist uniquely (finite or infinite).

Properties (without proof)

1. For any sequence $\{u_n\}$,

$$\inf u_n \le \underline{\lim}\, u_n \le \overline{\lim}\, u_n \le \sup u_n$$

2. For any sequence $\{u_n\}$,

$$\underline{\lim}\,(-u_n) = -\overline{\lim}\, u_n,$$

$$\overline{\lim}\,(-u_n) = -\underline{\lim}\, u_n .$$

Examples :

1. Let us consider the sequence $\{u_n\}$, where $u_n = (-1)^n$.

Now inf $\{u_n, u_{n+1}, u_{n+2}, ...\}$

= inf $\{(-1)^n, (-1)^{n+1}, (-1)^{n+2}, ...\}$

= – 1

$\sup \{u_n, u_{n+1}, u_{n+2}, ...\}$

$= \sup \{(-1)^n, (-1)^{n+1}, (-1)^{n+2}, ...\}$

$= 1$

$\therefore \underline{\lim}\, u_n = \sup_n \inf \{(1)^n, (-1)^{n+1}, (-1)^{n+2}, ...\}$

$= -1$

$\overline{\lim}\, u_n = \inf \sup \{(-1)^n, (-1)^{n+1}, (-1)^{n+2}, ...\}$

$= 1$

2. Let us consider the sequence $\{u_n\}$, where $u_n = 1 + (-1)^n$

Now $\inf \{u_n, u_{n+1}, u_{n+2}, ...\}$

$= \inf \{1+(-1)^n, 1+(-1)^{n+1}, 1+(-1)^{n+2}, ...\}$

$= 0$

$\sup_n \{u_n, u_{n+1}, u_{n+2}, ...\}$

$= \sup_n \{1+(-1)^n, 1+(-1)^{n+1}, 1+(-1)^{n+2}, ...\}$

$= 2$

$\therefore \underline{\lim}\, u_n = 0, \overline{\lim}\, u_n = 2.$

3. Let us consider the sequence $\{u_n\}$, where $u_n = \frac{(-1)^n}{n^2}$

Now $\inf \{u_n, u_{n+1}, u_{n+2}, ...\}$

$= \inf \{\frac{(-1)^n}{n^2}, \frac{(-1)^{n+1}}{(n+1)^2}, \frac{(-1)^{n+2}}{(n+2)^2}, ...\}$

$$= \begin{cases} \frac{-1}{n^2}, \text{if } n \text{ is odd} \\ \frac{-1}{(n+1)^2}, \text{if } n \text{ is even} \end{cases}$$

$$\therefore \underline{\lim}\, u_n = \sup_n \inf \left\{\frac{(-1)^n}{n^2}, \frac{(-1)^{n+1}}{(n+1)^2}, \frac{(-1)^{n+2}}{(n+2)^2}, ...\right\}$$

$= 0$

Similarly, $\overline{\lim}\, u_n = 0$

4. Let us consider the sequence $\{u_n\}$,

where $u_n = n$

Now $\inf \{u_n, u_{n+1}, u_{n+2}, ...\}$

$= \inf \{u, n+1, n+2, ...\}$

=n

$\therefore \underline{\lim}\, u_n = +\infty$

Similarly, $\overline{\lim}\, u_n = +\infty$

5. Let us consider the sequence $\{u_n\}$,

where $u_n = (-1)^n (1 + \frac{1}{n})$

Now $\inf \{u_n, u_{n+1}, u_{n+2}, ...\}$

$$= \inf \left\{ (-1)^n \left(1+\frac{1}{n}\right), (-1)^{n+1} \left(1+\frac{1}{n+1}\right), (-1)^{n+2} \left(1+\frac{1}{n+2}\right), ... \right\}$$

$$= \begin{cases} -\left(1+\frac{1}{n}\right), \text{if } n \text{ is odd} \\ -\left(1+\frac{1}{n+1}\right), \text{if } n \text{ is even} \end{cases}$$

$$\underline{\lim}\, u_n = \sup_n \inf \left\{ (-1)^n \left(1+\frac{1}{n}\right), (-1)^{n+1} \left(1+\frac{1}{n+1}\right), ... \right\}$$

= –1

Similarly $\overline{\lim}\, u_n = 1$



3.9 LET US SUM UP

- A function f from the set N of natural numbers to the set R of real numbers is called a (real) sequence.
- A sequence $\{u_n\}$ is said to be bounded above if there exists a number M such that $u_n \leq M, \forall n \in N$ A sequence $\{u_n\}$ is said to be bounded below if there exists a number m such that $u_n \geq m, \forall n \in N$ A sequence is said to be bounded if it is both bounded above and bounded below.
- Let $\{u_n\}$ be a sequence. $\{u_n\}$ is said to be a monotomically decreasing sequence if

  $u_1 \geq u_2 \geq u_3 \geq u_4 \geq .....$

$\{u_n\}$ is said to be a monotomically increasing sequence if

$u_1 \le u_2 \le u_3 \le u_4 \le .....$

A sequence which is either monotonically increasing or monotonically decreasing is called a monotonic sequence.

- A sequence $\{u_n\}$ is said to be convergent if it has a limit $L \in R$.

  A sequence is said to be divergent if it is not convergent.
- A convergent sequence is bounded.
- Let $\{[a_n, b_n]\}$ be a sequence of closed and bounded intervals such that

  i) $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for all $n \in N$

  ii) $\lim (b_n - a_n) = 0$

  Then $\bigcap_{n=1}^{\infty} [a_n - b_n]$ contains exactly one point
- A necessary and sufficient condition for the convergence of a sequence $\{u_n\}$ is that for every $\varepsilon > 0$, there exists a positive integer m such that $|u_n - u_k| < \varepsilon$, for n, k > m (n > k).
- A number p is said to be a limit point of a sequence $\{u_n\}$ if, given any nbd of p, u_n belongs to the same for an infinite number of values of n. Thus, p is a limit point of $\{u_n\}$ if and only if given any positive number δ,

  $u_n \in]\, p-\delta, p+\delta\, [$ for an infinite number of values of n.



3.10 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress1

Ans to Q No 1: Intuitively, we have

$$\lim_{n \to \infty} \frac{1}{n^p} = 0, \quad p > 0$$

We are to prove that, given any $\varepsilon > 0$, there exists a positive integer m such that $\left| \frac{1}{n^p} = 0 \right| < \varepsilon$ for all $n \ge m$

Now $\left|\frac{1}{n^p}\right| < \varepsilon$, when $\frac{1}{n^p} < \varepsilon$

i.e. $n^p > \frac{1}{\varepsilon}$

i.e. $n > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}$, i.e.. $n > \frac{1}{\varepsilon^{\frac{1}{p}}}$

We take $m = \frac{1}{\varepsilon^{\frac{1}{p}}}$, which depends upon ε.

$\therefore \left|\frac{1}{n^p}\right| < \varepsilon,$ for all $n \geq m$.

$\therefore \lim_{n \to \infty} \frac{1}{n^p} = 0$

$\therefore$ The sequence converges to 0.

Ans to Q No 2: Try yourself.

The give sequence is convergent. It converges to 2.

Check Your Progress 2

Ans to Q No 1: Given that the sequence $\{u_n\}$ converges to L.

$\therefore$ given $\varepsilon > 0$, there exists a positive integer m such that

$|u_n - L| < \varepsilon$, for all $n \geq m$

Now $\big||u_n| - |L|\big| < |u_n - L|$

$< \varepsilon$, for all $n \geq m$

$\therefore$ The sequence $\{|u_n|\}$ converges to |L|.

The sequence $\{|u_n|\}$ is convergent.

2nd part :

The converse of the above result is not convergent.

Let us consider $\{u_n\}$, where $u_n = (-1)^n$. i.e. the sequence in {–1, 1, –1, 1, ...} clearly $\{u_n\}$ isnot convergent.

Now the sequence $\{|u_n|\}$, i.e. {1, 1, 1, ...} converges to 1.

Ans to Q No 2: If possible, let L > B

We take ε = L – B

$\therefore \{u_n\}$ converges to L, there exists a positive integer m such that $|u_n - L| < \varepsilon$, for all $n \geq m$

or, L – ε <u_n < L+ ε, for all $n \geq m$

This $\Rightarrow$ B < u_n $\Rightarrow$ u_n>B, which is a contradiction.

$\therefore$ our assumption is wrong.

$\therefore$ L $\leq$ B

Ans to Q No 3: Given that the sequence $\{u_n\}$ converges to L.

Let $\{u_{n_r}\}$, i.e., $\{u_{n_1}, u_{n_2}, u_{n_3}, ...\}$ be a subsequence of $\{u_n\}$.

Here $n_1 < n_2 < n_3 <$ .....

Let $\varepsilon > 0$ be given

$\therefore$ There exists a positive integer m such that

$|u_n - L| < \varepsilon$, for all $n \geq m$

$\therefore$ $n_1 < n_2 < n_3 <$ ..... is an increasing sequence of natural numbers, there is a positive integer i such that n_i, n_{i+1}, n_{i+2},..... are all $n \geq m$

$\therefore |u_{n_r} - L| < \varepsilon$, for $r \geq i$

$\therefore \{u_{n_r}\}$ is convergent.

Check Your Progress 3

Ans to Q No 1: Let us consider the sequence $\{u_n\}$, where u_n = n, and the sequence $\{u_n\}$ where υ_n = –n.

Clearly the sequences $\{u_n\}$, and $\{\upsilon_n\}$ are divergent, whereas the sequence $\{u_n + \upsilon_n\}$, i.e. the sequence {0, 0, 0, ...} is convergent.

We note that $\{u_n + \upsilon_n\}$, i.e. {0, 0, 0, ...} is a constant sequence, and it converges to 0.

Ans to Q No 2: Let us consider the sequence $\{u_n\}$, where u_n = n, and the sequence $\{\upsilon_n\}$, where $\upsilon_n = \frac{1}{n}$.

Clearly, the sequence $\{u_n\}$ is divergent. On the other hand, $\{\upsilon_n\}$ is convergent, and it converges to 0.

Now the sequence $\{u_n \upsilon_n\}$, i.e. the sequence {1, 1, 1, ...} is convergent. It is a constant sequence, and it converges to 1.



3.11 FURTHER READING

1) Principles of Mathematical Analysis by Walter Rudin

2) Methods of real Analysis by Richand R. Goldbery

3) Mathematical Analysis by S. C. Malik and Savita Arora

4) A Course of Mathematical Analysis by shanti Narayan.



3.12 MODEL QUESTIONS

Q.1: Examine if the sequence $\{u_n\}$, where $u_n = 1 + \frac{(-1)^n}{n}$ is bounded.

Q.2: Examine if the sequence $\{u_n\}$, where $u_n = \sqrt{n+1} - \sqrt{n}$ is convergent.

Q.3: Show that a convergent sequence has only one limit point.

Q.4: Show that the sequence $\{u_n\}$, where $u_n = \left(1 + \frac{1}{n}\right)^n$ is convergent.

Q.5: Prove that a convergent sequence is bounded.

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